APPENDIX I

To determine the solvability condition for a problem of the form

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Gamma_j}{d\rho}\right)+\left(\gamma_{mj}^2-\frac{m^2}{\rho^2}\right)\Gamma_j=F_j(\rho) \tag{61}$$

$$\Gamma_j(1)=c_j \tag{62}$$

we multiply (61) by a function $\rho u(\rho)$, to be specified later, integrate the result by parts from $\rho=0$ to $\rho=1$, and obtain

$$\int_0^1 \rho\Gamma_j\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\left(\gamma_{mj}^2-\frac{m^2}{\rho^2}\right)u\right]d\rho + u(1)\frac{d\Gamma_j}{d\rho}(1)-\Gamma_j(1)\frac{du}{d\rho}(1) = \int_0^1 \rho u(\rho)F_j(\rho)\,d\rho. \tag{63}$$

We choose $u(\rho)$ to be a solution of the so-called adjoint homogeneous problem

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)+\left(\gamma_{mj}^2-\frac{m^2}{\rho^2}\right)u=0 \tag{64}$$

$$u(1)=0. \tag{65}$$

We take the solution of (64) and (65) that is bounded at $\rho=0$ as $u(\rho)=J_m(\gamma_{mj}\rho)$. Substituting for u into (63) and using the boundary condition (62), we arrive at the following solvability condition:

$$c_j\gamma_{mj}J_m'(\gamma_{mj})+\int_0^1 \rho J_m(\gamma_{mj}\rho)F_j(\rho)\,d\rho=0. \tag{66}$$

APPENDIX II

$$E_{0j}=-\tfrac{1}{2}\gamma_{mj}^4J_m''(\gamma_{mj})+\tfrac{1}{4}\gamma_{mj}J_m'(\gamma_{mj})\cdot\{(\gamma_{mj}^2-k_jk_w)^2[J_m'(\alpha_j)/\alpha_jJ_m(\alpha_j)] + (\gamma_{mj}^2+k_jk_w)^2[J_m'(\beta_j)/\beta_jJ_m(\beta_j)]\}$$

$$E_{1j}=\tfrac{1}{4}(\gamma_{mj}^2-k_jk_w)\{\gamma_{mj}^2J_m''(\gamma_{mj})-(\gamma_{mj}/\alpha_j)\cdot(\gamma_{mj}^2-2k_w^2-3k_jk_w)[J_m'(\gamma_{mj})J_m'(\alpha_j)/J_m(\alpha_j)]\}$$

$$E_{2j}=\tfrac{1}{4}(\gamma_{mj}^2+k_jk_w)\{\gamma_{mj}^2J_m''(\gamma_{mj})-(\gamma_{mj}/\beta_j)\cdot(\gamma_{mj}^2-2k_w^2+3k_jk_w)[J_m'(\gamma_{mj})J_m'(\beta_j)/J_m(\beta_j)]\}$$

$$D_{0n}=\tfrac{1}{4}\gamma_{mn}\{J_m'(\gamma_{mn})([k_s(\gamma_{mn}^2+k_nk_w)(2k_s+k_w) + (\gamma_{mn}^2-k_nk_w)^2[J_m'(\alpha_n)/\alpha_nJ_m(\alpha_n)]) - 2\gamma_{mn}^3J_m''(\gamma_{mn})\}$$

$$D_{0s}=\tfrac{1}{4}\gamma_{ms}\{J_m'(\gamma_{ms})([k_n(\gamma_{ms}^2-k_sk_w)(2k_n-k_w) + (\gamma_{ms}^2+k_sk_w)^2[J_m'(\beta_s)/\beta_sJ_m(\beta_s)]) - 2\gamma_{ms}^3J_m''(\gamma_{ms})\}.$$

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Asymmetric Coupled Transmission Lines in an Inhomogeneous Medium

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***Abstract*—Terminal characteristic parameters for a uniform coupled-line four-port for the general case of an asymmetric, inhomogeneous system are derived in this paper. The parameters (impedance, admittance, etc.) are derived in terms of two independent modes that propagate in two uniformly coupled propagating systems. The four-port parameters derived are of the same form as those obtained for the symmetric case resulting in similar two-port equivalent circuits for various circuit configurations considered by Zysman and Johnson [1]. The results obtained should be quite useful in designing asymmetric coupled-line circuits in an inhomogeneous medium for various known applications.**

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INTRODUCTION

UNIFORM coupled-line circuits are used for many applications including filters, couplers, and impedance matching networks. These circuits are usually designed by utilizing the impedance, admittance, chain, and other parameters characterizing the coupled-line four-port network. These parameters may be obtained in terms of the coupled-line impedances or admittances, and phase velocities for even and odd modes of excitation for the case of coupled TEM lines (homogeneous medium) [2], [3] or coupled identical lines in an inhomogeneous medium [1]. Recalling that even and odd modes of excitation correspond to the cases where the voltages and the currents on the two lines are equal in magnitude and are in phase for the even mode and out of phase for the odd mode, it is seen that such modes cannot propagate independently for the case of asymmetric coupled lines [5]. For asymmetric coupled-line cases these modes can be defined only for special cases [5]–[7] where the line parameters obey certain restrictive relationships.

In this paper the parameters of a general asymmetric asynchronous coupled-line four-port are obtained in terms of the line properties for two independent modes of excitations. These modes correspond to a linear combination of voltages and currents on the two lines which are related in magnitude and phase through terms involving line constants. The four-port circuit parameters are obtained by writing the solutions for voltages and currents on the two lines in terms of the two independent modes and deriving the relationships between port voltages and currents in a suitable form leading to impedance, admittance, chain, or any other parameters.

COUPLED-LINE ANALYSIS

The behavior of two coupled lines is described in general by the following set of equations:

$$-\frac{dv_1}{dx} = z_1 i_1 + z_m i_2 \qquad (1a)$$

$$-\frac{dv_2}{dx} = z_2 i_2 + z_m i_1 \qquad (1b)$$

$$-\frac{di_1}{dx} = y_1 v_1 + y_m v_2 \qquad (2a)$$

$$-\frac{di_2}{dx} = y_2 v_2 + y_m v_1 \qquad (2b)$$

where $z_j (j = 1,2)$ and $y_j (j = 1,2)$ are self-impedance and admittance per unit length of line j in the presence of line k $(k = 1,2; k \neq j)$, z_m and y_m are mutual impedance and admittance per unit length, respectively, and an $e^{j\omega t}$ time variation has been assumed.

Differentiating (1a) and (1b) with respect to x and substituting (2a) and (2b), a system of equations for voltages on the uniformly coupled lines is obtained as

$$\frac{d^2 v_1}{dx^2} - a_1 v_1 - b_1 v_2 = 0 \qquad (3a)$$

$$\frac{d^2 v_2}{dx^2} - a_2 v_2 - b_2 v_1 = 0 \qquad (3b)$$

where

$$a_1 = y_1 z_1 + y_m z_m$$
$$a_2 = y_2 z_2 + y_m z_m$$
$$b_1 = z_1 y_m + y_2 z_m$$
$$b_2 = z_2 y_m + y_1 z_m. \qquad (4)$$

Since none of the coefficients in (3) varies with x, an x variation of the form $v(x) = v_0 e^{\gamma x}$ is assumed for the voltages. The solution of the resulting eigenvalue problem leads to the following four roots of γ:

$$\gamma_{1,2} = \pm \gamma_c$$

and

$$\gamma_{3,4} = \pm \gamma_\pi$$

where

$$\gamma_{c,\pi}^2 = \frac{a_1 + a_2}{2} \pm \tfrac{1}{2}[(a_1 - a_2)^2 + 4b_1 b_2]^{1/2}. \qquad (5)$$

For the case of lossless coupled systems these roots are the same as those obtained by Amemiya [8], Krage and Haddad [9], Marx [10], and others.

These values of γ_c and γ_π correspond to in phase and antiphase waves for a class of lossless lines. The relationship between the voltages on the two lines for each of these waves may be determined from (3) and (5) and is given as

$$\frac{v_2}{v_1} = \frac{\gamma^2 - a_1}{b_1} = \frac{b_2}{\gamma^2 - a_2} \qquad (6)$$

or

$$R_c \triangleq \frac{v_2}{v_1} \quad \text{for} \quad \gamma = \pm\gamma_c$$
$$= \frac{1}{2b_1}\{(a_2 - a_1) + [(a_2 - a_1)^2 + 4b_1 b_2]^{1/2}\} \qquad (7)$$

and

$$R_\pi \triangleq \frac{v_2}{v_1} \quad \text{for} \quad \gamma = \pm\gamma_\pi$$
$$= \frac{1}{2b_1}\{(a_2 - a_1) - [(a_2 - a_1)^2 + 4b_1 b_2]^{1/2}\}. \qquad (8)$$

As seen from the expressions for R_c and R_π, v_2/v_1 is positive real for one mode, and negative real for the other mode for a large class of lossless coupled-line systems where $b_1 b_2 > 0$. For the case of identical lines, $R_c = +1$ and $R_\pi = -1$ and the two modes correspond to the even and odd modes, respectively [4], and for homogeneous systems R_c and R_π correspond to lateral and diagonal excitations, respectively [11].

The general solutions for the voltages on the two lines in terms of all the four waves then are given by

$$v_1 = A_1 e^{-\gamma_c x} + A_2 e^{\gamma_c x} + A_3 e^{-\gamma_\pi x} + A_4 e^{\gamma_\pi x} \qquad (9)$$

$$v_2 = A_1R_ce^{-\gamma_c x} + A_2R_ce^{\gamma_c x} + A_3R_\pi e^{-\gamma_\pi x} + A_4R_\pi e^{\gamma_\pi x}. \quad (10)$$

The corresponding currents for all four waves are determined by substituting the expressions for voltages (9) and (10) into (1a) and (1b) leading to

$$i_1 = A_1Y_{c1}e^{-\gamma_c x} - A_2Y_{c1}e^{\gamma_c x} + A_3Y_{\pi 1}e^{-\gamma_\pi x} - A_4Y_{\pi 1}e^{\gamma_\pi x} \quad (11)$$

$$i_2 = A_1R_cY_{c2}e^{-\gamma_c x} - A_2R_cY_{c2}e^{\gamma_c x} + A_3R_\pi Y_{\pi 2}e^{-\gamma_\pi x} - A_4R_\pi Y_{\pi 2}e^{\gamma_\pi x} \quad (12)$$

where Y_{c1}, Y_{c2}, $Y_{\pi 1}$, and $Y_{\pi 2}$ are the characteristic admittances of lines 1 and 2 for the two modes and are given by

$$Y_{c1} = \gamma_c \frac{z_2 - z_mR_c}{z_1z_2 - z_m^2} = \frac{1}{Z_{c1}} \quad (13)$$

$$Y_{c2} = \frac{\gamma_c}{R_c}\frac{z_1R_c - z_m}{z_1z_2 - z_m^2} = \frac{1}{Z_{c2}} \quad (14)$$

$$Y_{\pi 1} = \gamma_\pi \frac{z_2 - z_mR_\pi}{z_1z_2 - z_m^2} = \frac{1}{Z_{\pi 1}} \quad (15)$$

$$Y_{\pi 2} = \frac{\gamma_\pi}{R_\pi}\frac{z_1R_\pi - z_m}{z_1z_2 - z_m^2} = \frac{1}{Z_{\pi 2}}. \quad (16)$$

From these equations and (7) and (8) for R_c and R_π, respectively, it is seen that

$$\begin{bmatrix} I_1 \\ I_2 \\ -I_3 \\ -I_4 \end{bmatrix} = \begin{bmatrix} Y_{c1} & -Y_{c1} & Y_{\pi 1} & -Y_{\pi 1} \\ R_cY_{c2} & -R_cY_{c2} & R_\pi Y_{\pi 2} & -R_\pi Y_{\pi 2} \\ R_cY_{c2}e^{-\gamma_c l} & -R_cY_{c2}e^{\gamma_c l} & R_\pi Y_{\pi 2}e^{-\gamma_\pi l} & -R_\pi Y_{\pi 2}e^{\gamma_\pi l} \\ Y_{c1}e^{-\gamma_c l} & -Y_{c1}e^{\gamma_c l} & Y_{\pi 1}e^{-\gamma_\pi l} & -Y_{\pi 1}e^{\gamma_\pi l} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (19)$$

$$\frac{Y_{c1}}{Y_{c2}} = \frac{Y_{\pi 1}}{Y_{\pi 2}} = -R_cR_\pi \quad (17)$$

and that the ratio of current amplitudes on the two lines are $i_2/i_1 = -1/R_\pi$ and $-1/R_c$ for the two modes $\gamma = \pm\gamma_c$ and $\gamma = \pm\gamma_\pi$, respectively.

Two independent modes can be excited on any two uniformly coupled systems. These modes correspond to a linear combination of voltages and currents which are related in magnitude and phase. The voltages and currents are related through $v_2/v_1 = R_c$ and R_π with $i_2/i_1 = -1/R_\pi$ and $-1/R_c$, respectively. This can be further illustrated from (1) and (2) by linearly combining the equations as $v_{c,\pi} = v_2 - R_{c,\pi}v_1$ and $i_{c,\pi} = i_2 + (1/R_{\pi,c})i_1$ resulting in uncoupled transmission-line equations for the two modes.

COUPLED-LINE FOUR-PORT

The impedance, admittance, or chain matrix for the coupled-line four-port as shown in Fig. 1 can now be obtained by solving for port current-voltage relationships

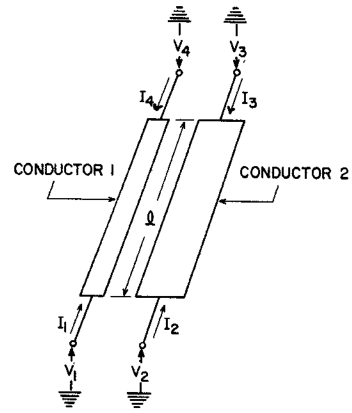


Fig. 1. Schematic of a uniform coupled-line four-port.

from (9)–(12). For example, the impedance matrix for the four-port is found by solving for port voltages in terms of port currents. The port voltages are given as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ R_c & R_c & R_\pi & R_\pi \\ R_ce^{-\gamma_c l} & R_ce^{\gamma_c l} & R_\pi e^{-\gamma_\pi l} & R_\pi e^{\gamma_\pi l} \\ e^{-\gamma_c l} & e^{\gamma_c l} & e^{-\gamma_\pi l} & e^{\gamma_\pi l} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}. \quad (18)$$

The port currents are given as

eliminating the amplitude coefficients A_1, A_2, A_3, and A_4 leads to four equations for V_1, V_2, V_3, and V_4 in terms of I_1, I_2, I_3, and I_4 of the form

$$[V] = [Z]\cdot[I]. \quad (20)$$

The elements of the 4×4 Z-matrix are given by

$$Z_{11} = Z_{44} = \frac{Z_{c1}\coth\gamma_c l}{(1 - R_c/R_\pi)} + \frac{Z_{\pi 1}\coth\gamma_\pi l}{(1 - R_\pi/R_c)} \quad (21a)$$

$$Z_{12} = Z_{21} = Z_{34} = Z_{43} = \frac{Z_{c1}R_c\coth\gamma_c l}{(1 - R_c/R_\pi)} + \frac{Z_{\pi 1}R_\pi\coth\gamma_\pi l}{(1 - R_\pi/R_c)}$$

$$= -\frac{Z_{c2}\coth\gamma_c l}{R_\pi(1 - R_c/R_\pi)} - \frac{Z_{\pi 2}\coth\gamma_\pi l}{R_c(1 - R_\pi/R_c)} \quad (21b)$$

$$Z_{13} = Z_{31} = Z_{24} = Z_{42} = \frac{R_cZ_{c1}}{(1 - R_c/R_\pi)\sinh\gamma_c l} + \frac{R_\pi Z_{\pi 1}}{(1 - R_\pi/R_c)\sinh\gamma_\pi l} \quad (21c)$$

$$Z_{14} = Z_{41} = \frac{Z_{c1}}{(1 - R_c/R_\pi)\sinh\gamma_c l} + \frac{Z_{\pi 1}}{(1 - R_\pi/R_c)\sinh\gamma_\pi l} \tag{21d}$$

$$Z_{22} = Z_{33} = -\frac{R_c Z_{c2}\coth\gamma_c l}{R_\pi(1 - R_c/R_\pi)} - \frac{R_\pi Z_{\pi 2}\coth\gamma_\pi l}{R_c(1 - R_\pi/R_c)}$$

$$= \frac{R_c^2 Z_{c1}\coth\gamma_c l}{(1 - R_c/R_\pi)} + \frac{R_\pi^2 Z_{\pi 1}\coth\gamma_\pi l}{(1 - R_\pi/R_c)} \tag{21e}$$

$$Z_{23} = Z_{32} = \frac{R_c^2 Z_{c1}}{(1 - R_c/R_\pi)\sinh\gamma_c l} + \frac{R_\pi^2 Z_{\pi 1}}{(1 - R_\pi/R_c)\sinh\gamma_\pi l}. \tag{21f}$$

The admittance parameters are found in a similar fashion and are given as

$$Y_{11} = Y_{44} = \frac{Y_{c1}\coth\gamma_c l}{(1 - R_c/R_\pi)} + \frac{Y_{\pi 1}\coth\gamma_\pi l}{(1 - R_\pi/R_c)} \tag{22a}$$

$$Y_{12} = Y_{21} = Y_{34} = Y_{43} = -\frac{Y_{c1}\coth\gamma_c l}{R_\pi(1 - R_c/R_\pi)} - \frac{Y_{\pi 1}\coth\gamma_\pi l}{R_c(1 - R_\pi/R_c)} \tag{22b}$$

$$Y_{13} = Y_{31} = Y_{24} = Y_{42} = \frac{Y_{c1}}{(R_\pi - R_c)\sinh\gamma_c l} + \frac{Y_{\pi 1}}{(R_c - R_\pi)\sinh\gamma_\pi l} \tag{22c}$$

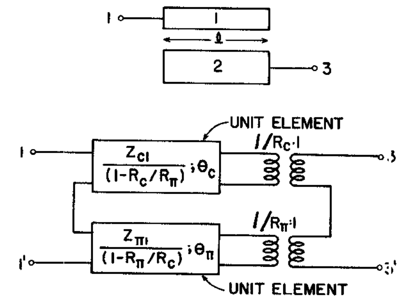
$$Y_{14} = Y_{41} = -\frac{Y_{c1}}{(1 - R_c/R_\pi)\sinh\gamma_c l} - \frac{Y_{\pi 1}}{(1 - R_\pi/R_c)\sinh\gamma_\pi l} \tag{22d}$$

$$Y_{22} = Y_{33} = -\frac{R_c Y_{c2}\coth\gamma_c l}{R_\pi(1 - R_c/R_\pi)} - \frac{R_\pi Y_{\pi 2}\coth\gamma_\pi l}{R_c(1 - R_\pi/R_c)} \tag{22e}$$

$$Y_{23} = Y_{32} = \frac{R_c Y_{c2}}{R_\pi(1 - R_c/R_\pi)\sinh\gamma_c l} + \frac{R_\pi Y_{\pi 2}}{R_c(1 - R_\pi/R_c)\sinh\gamma_\pi l}. \tag{22f}$$

TWO-PORT CIRCUITS

The parameters (matrix elements) characterizing a general uniform coupled-line four-port obtained previously are of the same form as those for the case of symmetric four-port derived by Zysman and Johnson [1]. The resulting equivalent circuits may be obtained in a similar fashion as in [1]. For example, for an open-circuit interdigital section consisting of lossless lines as shown in Fig. 2, $I_2 = I_4 = 0$ and



$$A = \frac{R_c^2 Z_{c1}(1-R_\pi/R_c)\cot\theta_c + R_\pi^2 Z_{\pi 1}(1-R_c/R_\pi)\cot\theta_\pi}{R_c Z_{c1}(1-R_\pi/R_c)\csc\theta_c + R_\pi Z_{\pi 1}(1-R_c/R_\pi)\csc\theta_\pi}$$

$$D = \frac{Z_{c1}(1-R_\pi/R_c)\cot\theta_c + Z_{\pi 1}(1-R_c/R_\pi)\cot\theta_\pi}{R_c Z_{c1}(1-R_\pi/R_c)\csc\theta_c + R_\pi Z_{\pi 1}(1-R_c/R_\pi)\csc\theta_\pi}$$

$$C = \frac{j(1-R_c/R_\pi)(1-R_\pi/R_c)}{R_c Z_{c1}(1-R_\pi/R_c)\csc\theta_c + R_\pi Z_{\pi 1}(1-R_c/R_\pi)\csc\theta_\pi}$$

$$B = \frac{AD-1}{C}$$

Fig. 2. Prototype open-circuited interdigital section.

$$\begin{bmatrix} V_1 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{13} \\ Z_{31} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_3 \end{bmatrix}. \tag{23}$$

Substituting for Z_{11}, Z_{13}, Z_{31}, and Z_{33} yields

$$\begin{bmatrix} Z_{11} & Z_{13} \\ Z_{31} & Z_{33} \end{bmatrix} = -j\frac{Z_{c1}}{(1 - R_c/R_\pi)}\begin{bmatrix} \cot\theta_c & R_c\csc\theta_c \\ R_c\csc\theta_c & R_c^2\cot\theta_c \end{bmatrix} - j\frac{Z_{\pi 1}}{(1 - R_\pi/R_c)}\begin{bmatrix} \cot\theta_\pi & R_\pi\csc\theta_\pi \\ R_\pi\csc\theta_\pi & R_\pi^2\cot\theta_\pi \end{bmatrix} \tag{24}$$

where $\theta_c = \beta_c l$ and $\theta_\pi = \beta_\pi l$.

This Z-matrix suggests an equivalent circuit as shown in Fig. 2 with its $ABCD$ parameters. The $ABCD$ parameters and the equivalent circuits for other configurations may be found in a similar manner. For the case of identical lines $R_c = -R_\pi = 1$ and the equivalent circuits and two-port parameters are the same as those obtained by Zysman and Johnson [1].

SPECIAL CASES

The results obtained above are indeed a generalized case of known results for various coupled-line systems where even- and odd-mode analysis has been applied. For various cases studied involving coupled TEM or inhomogeneous lines, the equations are simplified leading to the respective known results.

Case 1—Symmetric Coupled Lines [1], [2]

For this case $y_1 = y_2 = y$; $z_1 - z_2 = z$. Then $R_c = 1$, and $R_\pi = -1$.

Expressing y's and z's in terms of line constants, i.e., self- and mutual inductances and capacitances, it is seen that

$$Z_{c2} = Z_{c1} = Z_{0e} \quad \text{the even-mode impedance}$$

and

$$Z_{\pi 2} = Z_{\pi 1} = Z_{0o} \quad \text{the odd-mode impedance}$$

with

$$\gamma_{c,\pi} = [(y \pm y_m)(z \pm z_m)]^{1/2} = \gamma_{e,o}[4] \tag{25}$$

and the resulting expressions for the coupled-line four-port parameters are the same as those in Zysman and Johnson [1] for an inhomogeneous medium and Jones and Bolljahn [2] for a homogeneous medium (for TEM case $y_m/y = -z_m/z$).

Case 2—Asymmetric Coupled Lines in a Homogeneous Medium [3], [6]

For lines with TEM waves

$$y_1 z_1 = y_2 z_2$$

and

$$\frac{y_m}{(y_1 y_2)^{1/2}} = -\frac{z_m}{(z_1 z_2)^{1/2}}. \tag{26}$$

Then

$$\gamma_c = \gamma_\pi = j\beta \tag{27}$$

$$R_c = -R_\pi = (Z_2/Z_1)^{1/2} \tag{28}$$

where $Z_1 = (z_1/y_1)^{1/2}$ and $Z_2 = (z_2/y_2)^{1/2}$.

The resulting expressions for the coupled-line four-port are the same as those in [3] and [6]. For example, the impedance parameters are [from (21a)–(21f)]

$$Z_{11} = Z_{44} = -j/2(Z_1/Z_2)^{1/2}(Z_c + Z_\pi)\cot\theta \tag{29a}$$

$$Z_{12} = Z_{21} = Z_{34} = Z_{43} = -j/2(Z_c - Z_\pi)\cot\theta \tag{29b}$$

$$Z_{13} = Z_{31} = Z_{24} = Z_{42} = -j/2(Z_c - Z_\pi)\csc\theta \tag{29c}$$

$$Z_{14} = Z_{41} = -j/2(Z_1/Z_2)^{1/2}(Z_c + Z_\pi)\csc\theta \tag{29d}$$

$$Z_{22} = Z_{33} = -j/2(Z_2/Z_1)^{1/2}(Z_c + Z_\pi)\cot\theta \tag{29e}$$

$$Z_{23} = Z_{32} = -j/2(Z_2/Z_1)^{1/2}(Z_c + Z_\pi)\csc\theta \tag{29f}$$

where

$$Z_{c,\pi} = (Z_1 Z_2)^{1/2}\left[\frac{1 \pm y_m/(y_1 y_2)^{1/2}}{1 \mp y_m/(y_1 y_2)^{1/2}}\right]^{1/2}. \tag{30}$$

Examination of Z_c and Z_π in terms of line constants reveals that the even- and odd-mode impedances of the two lines as defined by Z_{0e}^a, Z_{0o}^a for line 1 and Z_{0e}^b and Z_{0o}^b for line 2, respectively, [6] are given by

$$Z_{0e}^a + Z_{0o}^a = (Z_1/Z_2)^{1/2}(Z_c + Z_\pi) \tag{31a}$$

$$Z_{0e}^a - Z_{0o}^a = Z_{0e}^b - Z_{0o}^b = Z_c - Z_\pi \tag{31b}$$

and

$$Z_{0e}^b + Z_{0o}^b = (Z_2/Z_1)^{1/2}(Z_c + Z_\pi). \tag{31c}$$

Case 3—A Congruent Case [5]

If the line constants are such that

$$\frac{y_1 + y_m}{y_2 + y_m} = \frac{z_2 - z_m}{z_1 - z_m} \triangleq R_3 \tag{32}$$

which is approximately the case for tightly coupled lines, the even and odd modes can be redefined as in [5]. Substitution of (32) into expressions for R_c and R_π, (7) and (8), leads to

$$R_c = +1$$

and

$$R_\pi = -\frac{y_1 + y_m}{y_2 + y_m} = -R_3. \tag{33}$$

The corresponding ratio of currents on the two lines is then given as

$$\frac{i_2}{i_1} = \frac{1}{R_3} \quad \text{for} \quad \gamma = \pm\gamma_c$$

and

$$\frac{i_2}{i_1} = -1 \quad \text{for} \quad \gamma = \pm\gamma_\pi. \tag{34}$$

Equations (33) and (34) correspond to the even- and odd-mode definitions for the coupled-line case where the condition given by (32) is satisfied [5]. Then the resulting matrix parameters are the same as those obtained by Speciale. These mode definitions have, of course, been experimentally verified for structures consisting of tightly coupled inhomogeneous lines.

CONCLUSIONS

It is shown that asymmetric, uniform coupled lines in an inhomogeneous medium, e.g., suspended substrate, microstrip lines, and others, may be analyzed in terms of the line properties for two independent modes of excitation. The mode characteristics, i.e., the propagation constants and the characteristic impedances, are derived in terms of the series impedances, the shunt admittances, and the mutual impedance and admittance per unit length of the lines. The 4 × 4 network matrices are then obtained in terms of these mode parameters. These circuit parameters characterizing the coupled-line four-port may be used to design various structures for all known applications including filters, couplers, and matching networks.

It should be noted that such structures can be treated utilizing the coupled-mode formulation [9]. However, the four-port circuit matrix is much easier and more convenient to use in formulating design procedures for various circuits particularly for the cases where multiple coupled-line sections are used. This paper has been primarily concerned with the study of inhomogeneous, asymmetric coupled lines. However, the formulation basically involves the evaluation of the properties of two linear uniformly coupled systems and coupled-line four-ports in terms of normal independent modes of the system and should provide a useful alternate tool for the study of many active and passive systems which have been studied using the coupled-mode theory.

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Long-Wavelength Electromagnetic Power Absorption in Prolate Spheroidal Models of Man and Animals

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Abstract—**A previously developed electromagnetic (EM) field perturbation analysis is used to calculate the electric fields in tissue prolate spheroids irradiated by plane waves with long wavelength compared to the spheroid dimensions. This theory is applied to prolate spheroid models of man and animals to obtain internal electric field strength, absorbed power distribution, and total absorbed power. These data are of value in estimating tissue EM power absorption in experimental animals and man. The theory may be used to help extrapolate animal biological effects data to man, and as a guide to establishing an EM radiation safety standard.**

INTRODUCTION

AN important aspect of electromagnetic- (EM) wave biological-effects research involves the investigation of internal electric field strength and power absorption in biological tissue subjected to EM irradiation. EM power is absorbed by the tissues as a function of frequency, body shape, tissue properties, and irradiation conditions. Absorbed power increases as the square of frequency at long wavelengths, enters a transition region of maximum absorbed power when the wavelength approximates body dimensions, and then decreases with frequency due to skin-effect surface heating. This general behavior has been characterized by Johnson and Guy [1] for a tissue sphere model, and has been measured experimentally by Gandhi [2] in irradiation experiments with rats.

Early work on the tissue sphere model has been done by Anne *et al.* [3], Shapiro *et al.* [4], Kritikos and Schwan [5], and Johnson and Guy [1]. Recent analyses of multilayer effects in spherical models have been reported by Joines and Spiegel [6], and Weil [7]. The principal result of the multilayer model compared to the homogeneous model is a shift in resonant frequency and an increase of peak absorption. These theoretical approaches are applicable to all frequency ranges and require extensive computer computations. Simpler low-frequency Mie solutions have been obtained by Lin *et al.* [8].

A field perturbation approach has recently been developed and applied to prolate spheroid models for low ka values well below the maximum absorption frequency range [9]. A principal conclusion from the prolate spheroid results is that orientation of the body with respect to the incident plane-wave vectors is an extremely important variable which can make an order-of-magnitude difference in EM power absorption.

Considerable effort has also been expended to measure

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